Асимптотики в задачах о набеге на берег длинных волн, порожденных локализованными источниками

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MOTIVATION:

PISTON MODEL for TSUNAMI PROBLEM

2-D SHALLOW WATER EQUATION

\[
\frac{\partial \eta}{\partial t} + \text{div}((\eta + D(x))u) = 0, \quad \frac{\partial u}{\partial t} + (u, \nabla)u + g\nabla \eta = 0.
\]

\[
u|_{t=0} = u^0\left(\frac{x}{\mu}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \eta|_{t=0} = \eta^0\left(\frac{x}{\mu}\right)
\]

LINEARIZATION in the OPEN OCEAN:

\[\eta \ll D(x), \quad |u| \ll \sqrt{gD(x)} \implies \]

\[
\frac{\partial \eta}{\partial t} + \text{div}(D(x)u) = 0, \quad \frac{\partial u}{\partial t} + g\nabla \eta = 0.
\]

The main mathematical problems: focal points and profile metamorphosis
BEAUTIFUL FUNDAMENTAL EXAMPLE

Let \[ V^0(y) = \left(1 + \left(\frac{y_1}{b_1}\right)^2 + \left(\frac{y_2}{b_2}\right)^2\right)^{-3/2}, \]

\[ \mathcal{P}(k_1, k_2) \] be a polynomial and \[ \hat{\mathcal{P}} = \mathcal{P}\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right). \]

WE SET: \[ V(y) = (\hat{\mathcal{P}}V^0)(T(\theta)y), \quad T(\theta) = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \theta & \cos \theta \end{pmatrix}, \]

AND \[ \eta|_{t=0} = V\left(\frac{x}{\mu}\right) \]
General


Global asymptotics without run-up

Run-up


В. А. Фок, “О каноническом преобразовании в классической и квантовой механике” *Вестн. Ленинградск. ун-та*, 1959, № 16, 67–70.


1-D SHALLOW WATER EQUATION OVER NONUNIFORM BOTTOM WITH THE DEPTH \( D(x) \)

\[
\eta_t + \frac{\partial}{\partial x}[v(\eta + D(x))] = 0, \quad v_t + vv_x + g\eta_x = 0
\]

\( \eta(x, t) \rightarrow \) free elevation

\( v(x, t) \rightarrow \) velocity

\( D(x) = \gamma^2 x + O(x^2) \rightarrow \) depth \quad \( c^2(x) = gD(x) \)

CAUCHY PROBLEM:

\[
\eta|_{t=0} = e(x)V\left(\frac{x-a}{\mu}\right), \quad v|_{t=0} = 0
\]
CARIER-GREENSPAN TRANSFORM
THE LINEAR WAVE EQUATION for $N(\tau, y), U(\tau, y)$:

$$N_\tau + \frac{\partial}{\partial y}(\gamma^2 y U) = 0, \quad U_\tau + g N_y = 0,$$

CONSIDER the SYSTEM

$$x = y + N(y, \tau) - \frac{1}{2} U_i^2(y, \tau), \quad t = \tau - U(y, \tau)$$

Let it defines one-to-one map from $\{y \geq 0, \tau \in \mathbb{R}\}$ to the value area of the right hand side

THEN

$$\eta = N - \frac{1}{2} U^2, \quad v = U$$

are the solution to the ORIGINAL NONLINEAR SYSTEM
in a PARAMETRIC FORM

$$\eta_t + \frac{\partial}{\partial x}[v(\eta + \gamma x)] = 0, \quad v_t + vv_x + g\eta_x = 0$$

CONCLUSION: WE NEED TO CONSTRUCT SOLUTIONS TO THE LINEAR WAVE EQUATION WITH the DEGENERATED VELOCITY NEAR the FOCAL POINT $x = 0$
THE EXACT SOLUTIONS WITHOUT PARAMETERS:

\[ N(y, \tau) = A \text{Re} \frac{(1 + i)(\tau + ib)}{(y - (\tau + ib)^2/4)^{3/2}}, \quad b > 0 \]

\[ V(y, \tau) = 2A \text{Re} \frac{1 + i}{(y - (\tau + ib)^2/4)^{3/2}}, \quad b > 0 \]

CREATION OPERATORS: THE POLYNOMIAL \( P(k) \quad \implies \)

THE OPERATOR \( P\left(\frac{\partial}{\partial \tau}\right) \quad \implies \)

NEW EXACT SOLUTIONS

\[ N_P = P\left(\frac{\partial}{\partial \tau}\right) N(y, \tau), \quad V_P = P\left(\frac{\partial}{\partial \tau}\right) V(y, \tau) \]


Linear and nonlinear interaction with the focal point (shore): jump of the Maslov index, the Hilbert transform, the profile metamorphosis and the creation of the “N-wave” (smoothed Dirac-δ function \(\rightarrow\) smoothed \(1/x\)- Sokhotskiy function)
The splash ("The physical level of rigor")

(Generalization of of Pelinovskii-Mazova idea to 2-D case if the source is far from the shore)

$x_1$ is the coordinate along the normal to the coast

$x_2$ is the coordinate along the coast

We fix $x_2$ and make the Carrier–Greenspan transform

$$\tilde{\eta} = \eta(x_1, x_2) - \frac{1}{2} u^2(x_1, x_2, t), \quad \tilde{x}_1 = x_1 + \tilde{\eta}, \quad \tilde{t} = t - u^2(x_1, x_2, t),$$

whence we find $\tilde{\eta} = \tilde{\eta}(x_1, x_2, t)$ and determine the uprush by solving the equation

$$\tilde{\eta}(\tilde{x}_1, x_2, t) + D(\tilde{x}_1, x_2) = 0.$$ 

$$D \approx \gamma x_1$$

$$\tilde{x}_1 \approx -\frac{1}{\gamma} \tilde{\eta}(\tilde{x}_1, x_2, t)$$

Hence one should study the 2-D linear problem for long waves
Linearization of free-boundary problem for the water waves

**Wave equation with localized source** (tsunami waves)

\[ \Omega \subset \mathbb{R}^2 \text{ a domain;} \]
\[ \partial \Omega \text{ smooth } \quad c(x) = \sqrt{gD(x)} \]

wave propagation in \( \Omega \)
from a source localized near \( x_0 \in \Omega \):

\[ \eta_{tt} - \langle \nabla, c^2(x) \nabla \rangle \eta = 0 \]
\[ \eta|_{t=0} = V(\mu^{-1}(x - x_0)), \quad \eta_t|_{t=0} = 0 \]

\( V(y) \in C^\infty \) decays at \( \infty \); \( \mu \to 0 \)
\[ c^2(x) \in C^\infty \}; \; c^2(x)|_{\partial \Omega} = 0; \]
\[ \nabla c^2(x)|_{\partial \Omega} \text{ vanishes nowhere} \]

\( \mu = \frac{\text{characteristic size of the source}}{\text{characteristic size of the basin}} \) is small

**Generalization**

\[ \eta_{tt} - \langle \nabla, c^2(x) \nabla \rangle \eta = F(x,t), \quad \eta|_{t=0} = 0, \quad \eta_t|_{t=0} = 0, \]
\[ F(x,t) = \lambda^2 g'(\lambda t)(\text{eV}) \left( \frac{x - x_0}{\mu} \right), \quad t \in [0,T], \]
“Missing” boundary conditions

No “classical” boundary conditions needed on $\partial \Omega$ owing to degeneration.

Oleinik & Radkevich 1969

Instead: Friedrichs extension of $-\langle \nabla, c^2(x) \nabla \rangle$

Energy integral

$$J^2(t) = \frac{1}{2} \left( \| \eta_t \|_{L^2}^2 + \| c(x) \nabla \eta \|_{L^2}^2 \right)$$

well-posedness (standard theorem)
The solution is localized near a front: moving boundary layer

Wave front at time $t$ is the projection onto $\mathbb{R}_x^2$ of the set of endpoints of trajectories of the Hamiltonian system

$$
\dot{x} = H_p, \quad \dot{p} = -H_x, \quad H(x, p) = c(x) \sqrt{p_1^2 + p_2^2}
$$

with the initial conditions

$$
x|_{t=0} = x_0, \quad p|_{t=0} = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix}
$$

$$
\psi \in [0, 2\pi)
$$

Solution: $p = P(\psi, t), x = X(\psi, t)$,

For fix $t$ the set $x = X(\psi, t)$ is the front in $\Omega$
The profile functions over Stoker round underwater bank: influence of the Morse (Maslov) index

Maslov index = 0

Maslov index = 1
The objects playing role in the description of the propagated wave:

i1) Fronts (bottom topography),

i2) the Green law \( \sqrt{4 \frac{D(x_0)}{D(x)}} \),

i3) the ray divergence \( \sqrt{\frac{1}{|X_\psi|}} \),

i4) the form of the initial source \( V \leftrightarrow \) the wave profile for caustics 1, 2)

i5) characteristic index of caustic for caustics 3)

i6) angle of the beach \( \frac{\partial c}{\partial n} \)
The wave field outside the focal points

\[ \eta \approx \sqrt{\frac{\mu}{|X_\psi(\psi, t)|}} \sqrt[4]{\frac{D(x_0)}{D(x)}} \text{Re}\left[ e^{-i\pi m(\psi, t)/2} F\left( \frac{y(x, t)}{\mu} \sqrt{\frac{D(x_0)}{D(x)}}, \psi \right) \right] \]

Here
\( y(x, t) \) is the alternative distance between the point \( x \) and the closest point \( X(\psi(x, t), t) \) on the front,
\( \psi(x, t) \) is the correspondence angle (coordinate) on the front,
\( m((\psi, t)) \) is the Morse (Maslov) index of this point.

\[ F(s, \psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_0^\infty \tilde{\eta}^0(\rho \mathbf{n}(\psi)) \sqrt{\rho} e^{i\rho} \, d\rho, \quad \tilde{\eta}^0(k) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \eta^0(z) e^{i(k, z)} \, dz, \quad \mathbf{n}(\psi) = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \]

\( \int_0^\infty (\cdot) \, d\rho \) gives the passage from fast oscillating functions to fast decaying ones.

Asymmetric Dotserenko-Sergievskii-Cherkasov-Wang type source

\[ \eta^0(z) = \frac{A}{(1 + (z_1/B_1)^2 + (z_2/B_2)^2)^{3/2}}, \quad F(s, \psi) = \frac{A e^{-i\frac{\pi}{4}}}{2\sqrt{2} \left( \sqrt{B_1^2 \cos^2 \psi + B_2^2 \sin^2 \psi} - is \right)^{3/2}}. \]

Caustics: 1) standard \( X_\psi = 0 \), 2) shore \( D(x) = 0 \)

\[ c(x) = 0 \]
The profile functions near the focal points

\[ \frac{\partial X}{\partial \psi} = 0, \quad \frac{\partial^2 X}{\partial \psi^2} = 0 \]

\[ \frac{\partial X}{\partial \psi} = 0, \quad \frac{\partial^2 X}{\partial \psi^2} \neq 0 \]
Standard focal points, cascade of time-space caustics, creation and dynamics of waves trapped by underwater banks and ridges

1) For $t = 0$ the front is the point $x = x_0$ (“non general position”)

2) For some $\tilde{t}$ and $\tilde{\psi}$ the vector $\frac{\partial X}{\partial \psi}(\tilde{\psi}, \tilde{t}) = 0$, the focal point $x = X(\tilde{\psi}, \tilde{t})$ belong to 2-D space-time caustic

(cf Berry and O’Nill in optics)
The wave train for 4 caustics
The boundary (shore) \( c(x) = 0 \) is the **caustic** of the special type!

\[
c(x) = \sqrt{gD(x)}
\]

\[
H = |p|c(x) = c(x_0) \quad \implies \quad |p| = \infty \quad \text{as} \quad c(x) = 0
\]

Change of variables: \( x = \text{coordinate along the normal} \)

Questions:
1) How to extend the trajectories after it hits the shore \( c(x) = 0 \)?
2) How to construct the wave field near the shore \( c(x) = 0 \)?
The (local) canonical change of variables \( p, x \to (\theta, \xi, q, y) \) near the shore \( x_1 = f(x_2), \ c(x) = \sqrt{x_1 - f(x_2)\gamma(x)}, \ \gamma(x) > 0 \) is smooth

\[
q = p_1^{-1}, \quad y = x_2, \quad \theta = p_1^2(x_1 - f(x_2)), \quad \xi = p_2 + f'(x_2)p_1, 
\]

\[
d\theta \wedge dq + d\xi \wedge dy = dp_1 \wedge dx_1 + dp_2 \wedge dx_2. 
\]

The Hamiltonian

\[
H = c(x)\sqrt{p_1^2 + p_2^2} = \sqrt{\theta} \gamma(q^2\theta + f(y), y) \sqrt{1 + (f'(y) - q\xi)^2}. 
\]

This gives the extension of the trajectories after their hits with the shore +

The trajectories are normal to the shore line!
\[ p > 0, \quad p(x) = 1/c(x) \]

\[ p < 0, \quad p(x) = -1/c(x) \]
The Fock quantization of the canonical transform:
for each canonical change of coordinates in the phase space one
can construct the unitary operator $\hat{G}$ transferring functions in the
new coordinates (positions) to the function in the old coordinates
(positions).

The Maslov idea: near the caustics one can construct the WKB-type solution
in the appropriate new coordinates in the phase space and then compensate this
canonical change of variables by acting this unitary operator.

The Fock unitary operator corresponding to canonical change of variables
$(q = p_1^{-1}, \quad y = x_2, \quad \theta = p_1^2(x_1 - f(x_2)), \quad \xi = p_2 + f'(x_2)p_1)$
is the **HANKEL transform**:

$$[K^h_A a](x) = \rho^{3/2} e^{i\pi \left(\frac{1}{2} - m\right)} \frac{i}{\sqrt{2\pi \mu^{3/2}}} \int \int J_0 \left(\frac{2\rho \sqrt{\theta(x_1 - f(x_2))}}{\mu}\right) e^{\frac{i}{\hbar} \Phi(x_2, \theta, q)} a(x_2, \theta, q, \rho) \frac{d\theta \ dq}{\sqrt{|\tilde{J}(\theta, q, x_2)|}}$$
The simplification of the solution: 1) change the Bessel function by its integral representation, 2) use the stationary phase method, 3) choose the special type of the source (Dotsenko-Sergievskii-Cherkesov-Wang == the Cauchy distribution)

\[ \eta|_{t=0} = W( Tx ) \]

\[ W(z) = \frac{A}{[1 + (z_1/l_1)^2 + (z_2/l_2)^2]^{3/2}}, \quad T = \begin{pmatrix} \cos \psi_0 & \sin \psi_0 \\ -\sin \psi_0 & \cos \psi_0 \end{pmatrix}, \quad l_1 \geq l_2 > 0, \]

Then we have in dimension variables (regular case)

\[ \eta \approx \frac{\sqrt{2D_0} Al_1 l_2}{\sqrt{\tg T} |X_{\psi}(\tau(\psi), \psi)|^{1/2}} \text{Re} \left[ \frac{e^{-im/2}(l(\psi) + i(c_0 t - \tau(\psi)))}{\left[ 4(x_1 - f(x_2))\Theta(\tau(\psi), \psi) + (l(\psi) + i(c_0 t - \tau(\psi)))^2 \right]^{3/2}} \right], \]

where \( l(\psi) = \sqrt{l_1^2 \cos^2(\psi - \psi_0) + l_2^2 \sin^2(\psi - \psi_0)}, \quad \psi = \psi(x_2) \)
The splash

Water

Source

Trajectory X(ψ,t)

Beach: x_i = f(x)

l_1

l_2

β

ψ

90°

\frac{-\eta_{max}}{\tan \theta}

\eta_{max}
Filtration of the front for Algerian tsunami
21 May 2003 Boumerdès–Zemmouri tsunami (Algeria)

<table>
<thead>
<tr>
<th>Place</th>
<th>Height actual</th>
<th>Height computed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carloforte</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>Imperia</td>
<td>0.08</td>
<td>0.12</td>
</tr>
<tr>
<td>Monaco</td>
<td>0.19</td>
<td>0.17</td>
</tr>
<tr>
<td>Nice</td>
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<td>0.06</td>
</tr>
<tr>
<td>Toulon</td>
<td>0.1</td>
<td>0.09</td>
</tr>
</tbody>
</table>

(The strike and eccentricity of the tsunami source have been taken from [A. Sahal et al., Nat. Hazards Earth Syst. Sci. 9, 823–1834, 2009]. The tsunami wave height at Carloforte was used to compute the only remaining source parameter, the amplitude $A$. The actual heights have been taken from the same paper.)
Conclusion: 1) We almost construct the base for the fast analytical-numerical algorithm for calculation of the runup of long waves generated by localised source.

2) Carrier-Grinspane transform gives a possibility to construct a lot of interesting exact solutions in the shallow water theory.

3) The Fock quantization of the canonical transform is not only interpretation, it is the real instrument.

THANK YOU FOR YOUR ATTENTION!
The trajectories in old and new coordinates

\[ x = X(\tau, \psi), \quad p = P(\tau, \psi), \quad X(0, \psi) = x_0, \quad P(0, \psi) = \mathbf{n}(\psi) = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \]

\[ q = Q(\tau, \psi), \quad y = Y(\tau, \psi), \quad \theta = \Theta(\tau, \psi), \quad \xi = \Xi(\tau, \psi). \]

Here \( \tau \) is the proper time, the fronts are \( \tau = t \).

The union of the trajectories \( \bigcup_{\psi, \tau} \) gives the 2-D (invariant) Lagrangian manifold \( \Lambda \) in the appropriate 4-D phase space.

The “abstract” global answer is the “\textit{twice}” modified Maslov canonical operator

\[ \int_0^\infty e^{-i \frac{c(0)t \rho}{\mu}} K_{\Lambda}^{-\mu/\rho} \eta^0(\rho \mathbf{n}(\psi)) d\rho. \]

The modification 1) is the integral over \( \rho \) (absolute value of the initial momenta) giving the passage from fast oscillating function to fast decaying ones and the modification 2) is presentation in the neighborhood of the shore.
The Fock quantization of the canonical transform:
for each canonical change of coordinates in the phase space one can construct the unitary operator $\hat{G}$ transferring functions in the new coordinates (positions) to the function in the old coordinates (positions).

The Maslov idea: near the caustics one can construct the WKB-type solution in the appropriate new coordinates in the phase space and then compensate this canonical change of variables by acting this unitary operator.

The old Maslov theory: in the neighborhood of standard caustic one pass from the coordinates $(p_1, p_2, x_1, x_2) \rightarrow (-x_1, p_2, p_1, x_2)$ and corresponding unitary operator is the partial FOURIER transform $F_{p_1 \rightarrow x_1}$.

$$[K^h a](x) \sim e^{\frac{i\pi}{2} (\frac{1}{2} - m)} \frac{\sqrt{\rho}}{\sqrt{2\pi \mu}} \int e^{i\rho (\Phi(p_1, x_2) - p_1 x_1)} \frac{a(x_2, p_1, \rho)}{\sqrt{|\tilde{J}(p_1, x_2)|}} dp_1,$$
Let \( a = l_2 / l_1 \) (eccentricity). Then on the coast we have

\[
\eta_{\text{max}} = \begin{cases} 
\frac{\sqrt{2D_0 A}}{\sqrt{\tan \theta |X_\psi|^1/2}} \frac{a}{\cos^2 (\psi - \beta) + a^2 \sin^2 (\psi - \beta)} \\
\frac{1}{8} \frac{\sqrt{2D_0 A}}{\sqrt{\tan \theta |X_\psi|^1/2}} \frac{a}{\cos^2 (\psi - \beta) + a^2 \sin^2 (\psi - \beta)} \\
\frac{9}{8\sqrt{3}} \frac{\sqrt{2D_0 A}}{\sqrt{\tan \theta |X_\psi|^1/2}} \frac{a}{\cos^2 (\psi - \beta) + a^2 \sin^2 (\psi - \beta)} 
\end{cases}
\]

for \( m = 4k \),

for \( m = 4k + 2 \),

for odd \( m \).